ISSM Workshop 2016

Ice Sheet System Model

Ice flow models

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Outline

1. Ice flow equations
   - Approximations implemented
   - Ice flow equation
   - Diagnostic parameters
   - Boundary conditions

2. Combining models
   - Methods implemented in ISSM
   - Penalties
   - Tiling method
   - Utilization

Ice Sheet flow equations

Incompressibility

$$\forall \boldsymbol{x} \in \Omega \qquad \nabla \cdot \boldsymbol{v} = \mathrm{Tr}\left(\dot{\boldsymbol{\varepsilon}}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

- $\boldsymbol{v} = (u, v, w)$ ice velocity (m/yr)
- $\dot{\boldsymbol{\varepsilon}}$ strain rate tensor (yr^{-1})

Incompressible viscous fluid

$$\boldsymbol{\sigma}' = 2\mu\dot{\boldsymbol{\varepsilon}} \tag{2}$$

- $\boldsymbol{\sigma}'$ deviatoric stress
- μ ice viscosity
- $\dot{\boldsymbol{\varepsilon}}$ strain rate tensor

Glen's flow law

$$\mu = \frac{B}{2\,\dot{\varepsilon}_e^{\frac{n-1}{n}}} \tag{3}$$

- B ice hardness
- n Glen's law coefficient ($n = 3$)
- $\dot{\varepsilon}_e$ effective strain rate (related to the second invariant)

Ice Sheet flow equations

Conservation of momentum

$$\forall \boldsymbol{x} \in \Omega \qquad \nabla \cdot \boldsymbol{\sigma}' - \nabla P + \rho \boldsymbol{g} = \boldsymbol{0} \tag{4}$$

Assumptions:

1. Stokes flow (quasi-static assumption)
2. Coriolis effect negligible

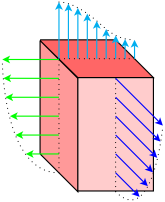
Boundary conditions

Ice/Air interface: Free surface	Γ_s	$\boldsymbol{\sigma} \cdot \boldsymbol{n} = P_{atm}\ \boldsymbol{n} \simeq \boldsymbol{0}$
Ice/Ocean interface: water pressure	Γ_w	$\boldsymbol{\sigma} \cdot \boldsymbol{n} = P_w\ \boldsymbol{n}$
Ice/Bedrock interface (1): lateral friction	Γ_b	$(\boldsymbol{\sigma} \cdot \boldsymbol{n} + \beta \boldsymbol{v})_{\parallel} = \boldsymbol{0}$
Ice/Bedrock interface (2): impenetrability	Γ_b	$\boldsymbol{v} \cdot \boldsymbol{n} = 0$
Side boundaries: Dirichlet	Γ_u	$\boldsymbol{v} = \boldsymbol{v}_{obs}$

Models description

Full-Stokes model (FS):

- Momentum balance + incompressibility
- 3D model
- Four unknowns (v_x, v_y, v_z, p)



Model equations

$$\left\{\begin{array}{l}
\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)-\frac{\partial p}{\partial x}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)-\frac{\partial p}{\partial y}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right)-\frac{\partial p}{\partial z}-\rho g=0
\end{array}\right.$$

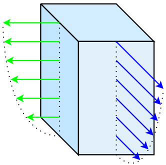
$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Models description

Higher-order model (HO):

- [Blatter, 1995, Pattyn, 2003]
- 3D model
- Horizontal and vertical velocity decoupled
- 2 (v_x, v_y) + 1 (v_z) unknowns

Blatter-Pattyn (BP)



(v_x, v_y)

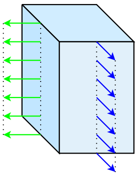
Model equations

$$\begin{cases} \frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y} + \mu\frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z} + \mu\frac{\partial v_z}{\partial x}\right) - \frac{\partial p}{\partial x} = 0 \\ \frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y} + \mu\frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z} + \mu\frac{\partial v_z}{\partial y}\right) - \frac{\partial p}{\partial y} = 0 \\ \frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z} + \mu\frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z} + \mu\frac{\partial v_z}{\partial y}\right) + \frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right) - \frac{\partial p}{\partial z} - \rho g = 0 \end{cases}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Models description

MacAyeal-Morland (SSA)



(v_x, v_y)

Shelfy-stream approximation (SSA):

- [MacAyeal, 1989]
- 2D model
- Horizontal and vertical velocity decoupled
- 2 (v_x, v_y) + 1 (v_z) unknowns

Model equations

$$\left\{\begin{array}{l}
\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)-\frac{\partial p}{\partial x}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)-\frac{\partial p}{\partial y}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right)-\frac{\partial p}{\partial z}-\rho g=0
\end{array}\right.$$

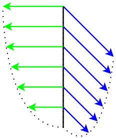
$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Models description

Hutter (SIA)

Shallow ice approximation (SIA):

- [Hutter, 1983]
- 3D analytical model
- 2 unknowns (v_x, v_y) computed separately



(v_x, v_y)

Model equations

$$\left\{\begin{array}{l}
\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)-\frac{\partial p}{\partial x}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)-\frac{\partial p}{\partial y}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right)-\frac{\partial p}{\partial z}-\rho g=0
\end{array}\right.$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Material non-linearity

Model equations

$$\left\{\begin{array}{l}
\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)-\frac{\partial p}{\partial x}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial y}+\mu\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)-\frac{\partial p}{\partial y}=0\\
\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial z}+\mu\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial z}+\mu\frac{\partial v_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial v_z}{\partial z}\right)-\frac{\partial p}{\partial z}-\rho g=0
\end{array}\right.$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Glen's flow law

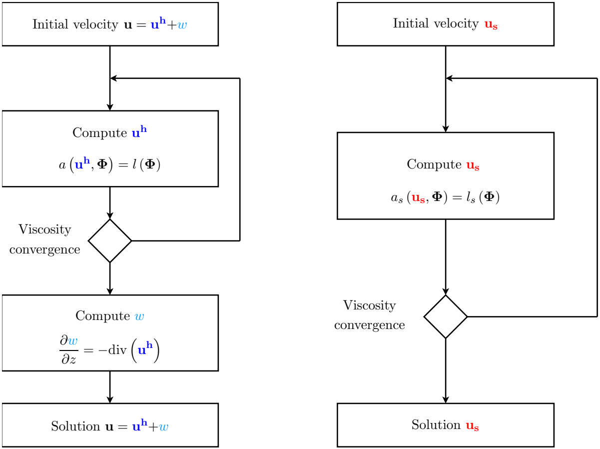
$$\mu=\frac{B}{2\,\dot{\varepsilon}_e^{\frac{n-1}{n}}} \tag{5}$$

- B ice hardness
- n Glen's law coefficient ($n=3$)
- $\dot{\varepsilon}_e$ effective strain rate (second invariant)

› Treatment of non-linearity with fixed point (Picard) or Newton's method

Material non-linearity

Treatment of non-linearity with fixed point:



Vertical velocity computed with incompressibility for SSA and HO

Flow equation

`setflowequation` is used to generate the approximation used to compute the velocity

- Arguments:
  1. model
  2. approximation names
  3. approximation domains
- Domains can be Argus files or array of element flags
- Approximation available
  - FS (Full-Stokes model)
  - HO (Higher-order model)
  - SSA (Shallow Shelf Approximation)
  - SIA (Shallow Ice Approximation)
- Possibility of coupling models

Flow equation

`setflowequation` is used to generate the approximation used to compute the velocity

- Examples

```
md=setflowequation(md,'SIA','all')
md=setflowequation(md,'FS','all')
md=setflowequation(md,'SSA','all')
md=setflowequation(md,'HO','all')
```

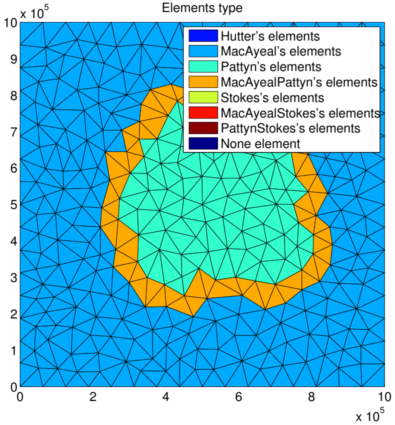
- To diplay the type of approximation:

```
>> plotmodel(md,'data','elements_type')
```

Flow equation

- To diplay the type of approximation:

```
>> plotmodel(md,'data','elements_type')
```



Flow equation class

```
>> md.flowequation

ans =

   flow equation parameters:
         isSIA                  : 0                 -- is the Shallow Ice Approximation (SIA) ?
         isSSA                  : 1                 -- is the Shelfy-Stream Approximation (SSA)?
         isL1L2                 : 0                 -- is the L1L2 approximation used ?
         isHO                   : 0                 -- is the Higher-Order (HO) approximation ?
         isFS                   : 0                 -- are the Full-FS (FS) equations used ?
         fe_SSA                 : 'P1'              -- Finite Element for SSA  'P1', 'P1bubble'...
         fe_HO                  : 'P1'              -- Finite Element for HO   'P1' 'P1bubble'...
         fe_FS                  : 'MINIcondensed'   -- Finite Element for FS   'MINIcondensed'...
         vertex_equation        : N/A               -- flow equation for each vertex
         element_equation       : N/A               -- flow equation for each element
         borderSSA              : N/A               -- vertices on SSA's border (for tiling)
         borderHO               : N/A               -- vertices on HO's border (for tiling)
         borderFS               : N/A               -- vertices on FS' border (for tiling)
```

StressBalance class

```
>> md.stressbalance

ans =

   StressBalance solution parameters:

      Convergence criteria:
         restol                  : 0.0001         -- mechanical equilibrium residual convergence criterion
         reltol                  : 0.01           -- velocity relative convergence criterion, NaN: not applied
         abstol                  : 10             -- velocity absolute convergence criterion, NaN: not applied
         isnewton                : 0              -- 0: Picard's fixed point, 1: Newton's method, 2: hybrid
         maxiter                 : 100            -- maximum number of nonlinear iterations
         viscosity_overshoot     : 0              -- over-shooting constant new=new+C*(new-old)

      boundary conditions:
         spcvx                   : N/A            -- x-axis velocity constraint (NaN means no constraint) [m/yr]
         spcvy                   : N/A            -- y-axis velocity constraint (NaN means no constraint) [m/yr]
         spcvz                   : N/A            -- z-axis velocity constraint (NaN means no constraint) [m/yr]

      Rift options:
         rift_penalty_threshold : 0              -- threshold for instability of mechanical constraints
         rift_penalty_lock      : 10             -- number of iterations before rift penalties are locked

      Penalty options:
         penalty_factor          : 3              -- offset used by penalties: penalty = Kmax*10^offset
         vertex_pairing          : N/A            -- pairs of vertices that are penalized

      Other:
         shelf_dampening         : 0              -- use dampening for floating ice ? Only for FS model
         FSreconditioning        : 10000000000000 -- multiplier for incompressibility equation. Only for FS model
         referential             : N/A            -- local referential
         loadingforce            : N/A            -- loading force applied on each point [N/m^3]
         requested_outputs       : {'default'}    -- additional outputs requested
```

Boundary conditions

Boundary conditions created automatically or manually

- Automatically:

```
>> md=SetIceSheetBC(md)
>> md=SetIceShelfBC(md,'Front.exp')
>> md=SetMarineIceSheefBC(md,'Front.exp')
```

- Manually: fields to change
  - `md.stressbalance.spcvx`
  - `md.stressbalance.spcvy`
  - `md.stressbalance.spcvz`
  - `md.mask.ice_levelset`

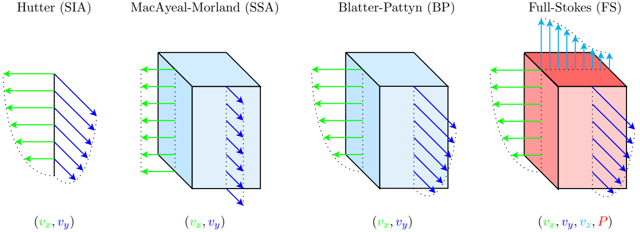
- To diplay the boundary conditions

```
>> plotmodel(md,'data','BC')
```

Models description

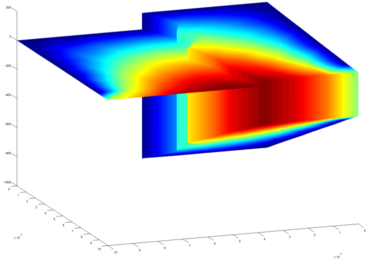
"Everything should be made as simple as possible, but no simpler." Albert Einstein

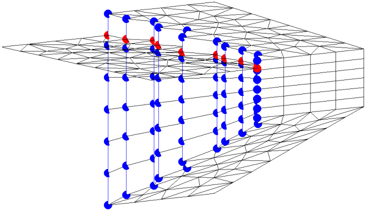
Model	Dim.	Unknowns	Reference
FS (Full-Stokes)	3d	4	[Stokes, 1845]
HO (Blatter-Pattyn)	3d	2 + 1	[Blatter, 1995, Pattyn, 2003]
SSA (Shallow shelf)	2d	2 + 1	[MacAyeal, 1989]
SIA (Shallow ice)	2d	2 + 1	[Hutter, 1983]



Penalty method

- Only to couple SSA and HO
- Very stiff spring to penalize differences between velocities at the interface



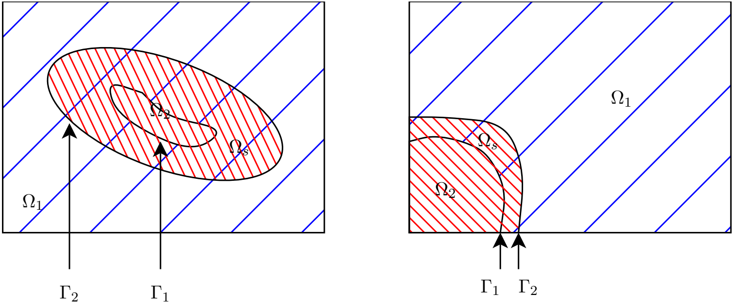


Using penalties to couple models:

```
md=setflowequation(md,'SSA','FloatingIce.exp','fill','HO','coupling','penalties')
```

Domain Decomposition

- $\Omega = \Omega_1 \cup \Omega_2$
- $\Omega_S = \Omega_1 \cap \Omega_2 \neq \varnothing$
- $\boldsymbol{u} = \boldsymbol{u_1}\,|_{\Omega_1} + \boldsymbol{u_2}\,|_{\Omega_2} \in \tilde{V}(\Omega) = (V_1(\Omega_1) + V_2(\Omega_2))$

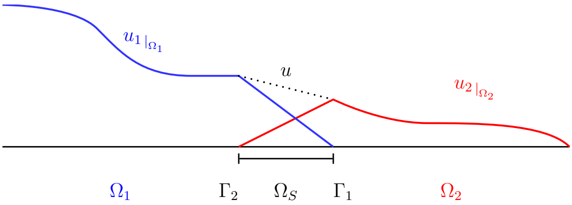


$$\text{Find } \boldsymbol{u} = \boldsymbol{u_1}\,|_{\Omega_1} + \boldsymbol{u_2}\,|_{\Omega_2} \in \tilde{V},$$

$$\forall(\boldsymbol{v_1}, \boldsymbol{v_2}) \in \tilde{V} \qquad a(\boldsymbol{u_1} + \boldsymbol{u_2}, \boldsymbol{v_1} + \boldsymbol{v_2}) = l(\boldsymbol{v_1} + \boldsymbol{v_2})$$

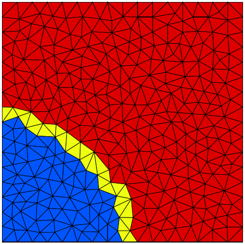
→ Infinite number of solutions for the continuous problem

Discretization



We take advantage of the discretization to avoid the redundancy:

$\rightarrow$ Create one layer of elements in the superposition zone



Multi-model formulation

Two different models: a_1, a_2 and l_1, l_2

Find $\boldsymbol{u} = \boldsymbol{u_1}|_{\Omega_1} + \boldsymbol{u_2}|_{\Omega_2} \in (V_1 + V_2)$, such that:

$$\forall \boldsymbol{v} = \boldsymbol{v_1}|_{\Omega_1} + \boldsymbol{v_2}|_{\Omega_2} \in (V_1 + V_2)$$

$$\underbrace{a_1\left(\boldsymbol{u_1}|_{\Omega_1}, \boldsymbol{v_1}|_{\Omega_1}\right)}_{\text{model 1}} + \underbrace{a_2\left(\boldsymbol{u_2}|_{\Omega_2}, \boldsymbol{v_2}|_{\Omega_2}\right)}_{\text{model 2}} +$$

$$\underbrace{a_2\left(\boldsymbol{u_1}|_{\Omega_1}, \boldsymbol{v_2}|_{\Omega_2}\right) + a_1\left(\boldsymbol{u_2}|_{\Omega_2}, \boldsymbol{v_1}|_{\Omega_1}\right)}_{\text{model coupling}}$$

$$= \underbrace{l_1\left(\boldsymbol{v_1}|_{\Omega_1}\right)}_{\text{model 1}} + \underbrace{l_2\left(\boldsymbol{v_2}|_{\Omega_2}\right)}_{\text{model 2}}$$

- Coupling different mechanical models
- Easy to implement (local modification of stiffness matrices)

Flow equation

`setflowequation` is used to generate the approximation used to compute the velocity

- Examples

```
md=setflowequation(md,'HO',md.mask.groundedice_levelset>0,'fill','SSA','coupling','penalties')
md=setflowequation(md,'HO',md.mask.groundedice_levelset>0,'fill','SSA','coupling','tiling')
md=setflowequation(md,'FS','Contour.exp','fill','HO')
```

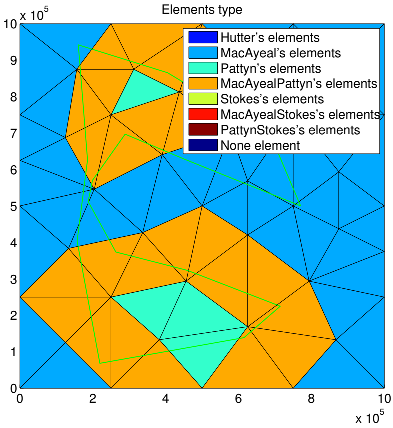
- Use `exptool` to create EXP contours

```
>> exptool('Contour.exp')
```

Flow equation

- To diplay the type of approximation:

```
>> plotmodel(md,'data','elements_type','edgecolor','k','expdisp','Contour.exp')
```



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Thanks!