Relativistic energy exchange between a polytrope and a Wyman IIa fluid

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Abstract

We present a theoretical analysis of the energy exchange between a relativistic polytropic fluid and an isotropic Wyman IIa fluid within a compact stellar object using the framework of gravitational decoupling by extended minimal geometric deformation (MGDe). By applying a mimic constraint for the energy density, we derive an effective anisotropic configuration that satisfies the main physical acceptability conditions for realistic stellar models. Our results show that the polytropic component transfers energy to the Wyman IIa fluid, predominantly in the outer layers of the star, while the core remains stable without energy exchange. The model confirms that physically viable configurations arise when the strong energy condition is preserved, offering insights into the internal dynamics and stability of relativistic multi-fluid systems.

1. Introduction

The **Theory of General Relativity (TRG)** [1, 2] provides the fundamental framework to describe the structure and dynamics of compact stellar objects such as neutron stars and black holes. Einstein's Field Equations (EFE)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \kappa\, T_{\mu\nu}, \tag{1}$$

link the geometry of spacetime, represented by the Einstein tensor $G_{\mu\nu}$, with the matter-energy distribution described by the energy–momentum tensor $T_{\mu\nu}$.
The interior of a compact stellar object is modeled by:

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\Omega^2, \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ represents the angular part of the metric.
The corresponding **effective matter sector** is obtained from Einstein's Field Equations and takes the form:

$$\kappa\rho = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \tag{3}$$

$$\kappa p_r = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{4}$$

$$\kappa p_t = -\frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right), \tag{5}$$

where ρ denotes the energy density, p_r the radial pressure, and p_t the tangential pressure inside the compact star.

2. Gravitational Decoupling and MGDe Framework

The **Gravitational Decoupling (GD)** via the **Extended Minimal Geometric Deformation (MGDe)** [5, 6, 7, 8] is based on the idea that the spacetime of a known seed source $T^{(s)}_{\mu\nu}$,

$$ds^2 = e^{\xi(r)}dt^2 - e^{\mu(r)}dr^2 - r^2 d\Omega^2, \tag{6}$$

is influenced by an additional source $\Theta_{\mu\nu}$ through metric deformations such that

$$\xi \to \nu = \xi + g(r), \tag{7}$$
$$e^{-\mu} \to e^{-\lambda} = e^{-\mu} + f(r), \tag{8}$$

where $f(r)$ and $g(r)$ are the **geometric deformation functions**, depending only on r to preserve spherical symmetry.
Substituting Eqs. (7)–(8) into Eqs. (3)–(5) leads to two distinct sets of differential equations:

(i) For the seed source $T^{(s)}_{\mu\nu}$:

$$\kappa T_0^{0(s)} = \frac{1}{r^2} - e^{-\mu}\left(\frac{1}{r^2} - \frac{\mu'}{r}\right), \tag{9}$$

$$\kappa T_1^{1(s)} = -\frac{1}{r^2} + e^{-\mu}\left(\frac{1}{r^2} + \frac{\xi'}{r}\right), \tag{10}$$

$$\kappa T_2^{2(s)} = \frac{e^{-\mu}}{4}\left(2\xi'' + \xi'^2 - \mu'\xi' + 2\frac{\xi' - \mu'}{r}\right). \tag{11}$$

(ii) For the additional source $\Theta_{\mu\nu}$:

$$\kappa\Theta_0^0 = -\frac{f}{r^2} - \frac{f'}{r}, \tag{12}$$

$$\kappa\Theta_1^1 - J_1 = f\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{13}$$

$$\kappa\Theta_2^2 - J_2 = \frac{f}{4}\left(2\nu'' + \nu'^2 + 2\frac{\nu'}{r}\right) - \frac{f'}{4}\left(\nu' + \frac{2}{r}\right), \tag{14}$$

where the auxiliary functions J_1 and J_2 are given by

$$J_1 = e^{-\mu}\frac{g'}{r}, \tag{15}$$

$$4J_2 = e^{-\mu}\left(2g'' + g'^2 + 2\frac{g'}{r} + 2\xi' g' - \mu' g'\right). \tag{16}$$

3. Wyman IIa Isotropic Fluid and Polytrope

Wyman IIa Isotropic Fluid

We consider as the seed source the **Wyman IIa isotropic fluid** [9], which its metric components are given by:

$$e^{\xi(r)} = (A - Br^2)^2, \tag{17}$$
$$e^{-\mu(r)} = 1 + Cr^2(A - 3Br^2)^{-2/3}, \tag{18}$$

where A is a dimensionless constant, while B and C have dimensions of inverse length squared.

Polytrope

Characterized by the equation of state [3]:

$$p_r = K\,\rho_\Theta^{1+\frac{1}{n}}, \tag{19}$$

where K is the polytropic constant, n the polytropic index, and ρ_Θ the energy density of the additional source.

The conditions for this case are: $B = 0.45$, $\kappa = 8\pi$, $R = 1$, along with

$$A = c_2\left(c_3 B R^2 - \frac{c_4\sqrt{R - 2M}}{\sqrt{R}}\right), \tag{20}$$

$$c = -\frac{2M\left(A - 3BR^2\right)^{2/3}}{R^3\left(\mathsf{K}^{\frac{1}{n}+1} + 1\right)}. \tag{21}$$

We use $n = 0.5$ to model neutron stars, and in the illustrations we observe that $K = 0.43$ corresponds to the **blue line**, $K = 0.44$ to the **black line**, and $K = 0.47$ to the **red line**.

4. Physical Acceptability Conditions [4] with Results

1. **Regular Spacetime:** The metric potentials of the interior solution must be positive, finite, and free of singularities throughout the star. At the center $(r = 0)$ they satisfy:

$$e^{-\lambda(0)} = 1, \quad e^{\nu(0)} = \text{constant}.$$

Moreover, $e^{-\lambda(r)}$ is a monotonically decreasing function, while $e^{\nu(r)}$ increases monotonically.

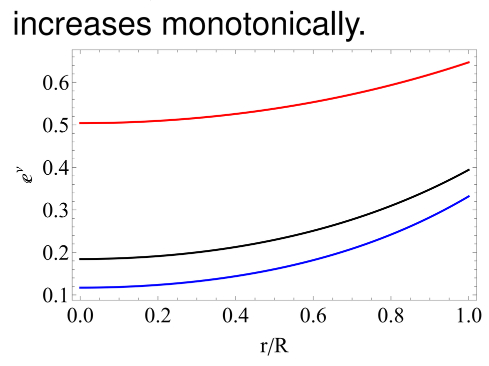

Figure 1: Temporal metric for different values of K

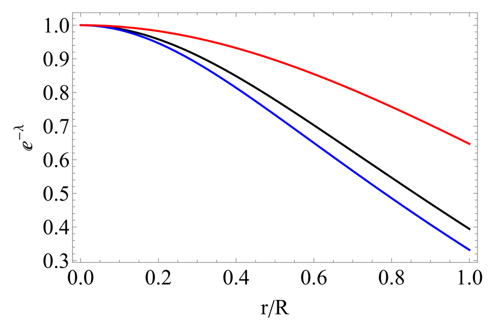

Figure 2: Radial metric for different values of K

2. **Matching Conditions:**

$$e^{\nu(R)} = e^{-\lambda(R)} = 1 - \frac{2M}{R}, \tag{22}$$

where M is the total mass and R the radius of the star. The radial pressure also vanishes at the surface:

$$p_r(R) = 0, \tag{23}$$

since the exterior region is vacuum.

3. **Causality Condition:**

$$0 \le v_r^2 = \frac{dp_r}{d\rho} < 1, \quad 0 \le v_t^2 = \frac{dp_t}{d\rho} < 1. \tag{24}$$

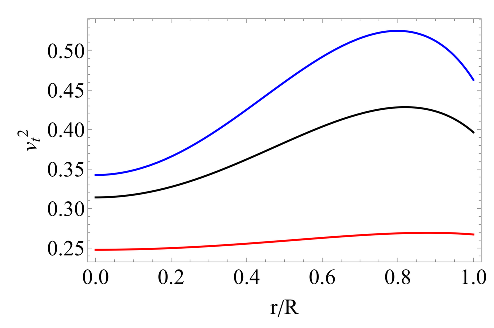

Figure 3: Temporal velocity for different values of K

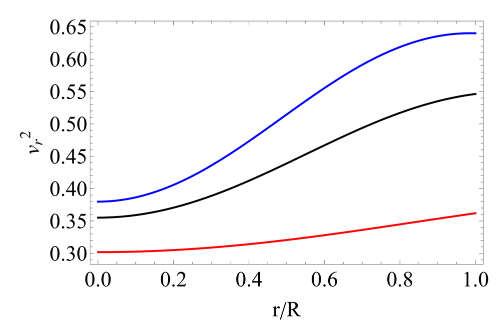

Figure 4: Radial velocity for different values of K

4. **Matter Sector:** The quantities ρ, p_r, and p_t must be positive, continuous, and monotonically decreasing functions of r.

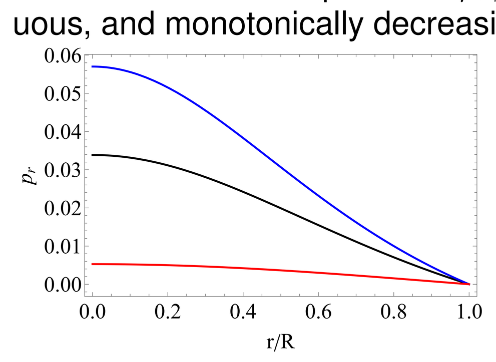

Figure 5: Radial pressure for different values of K

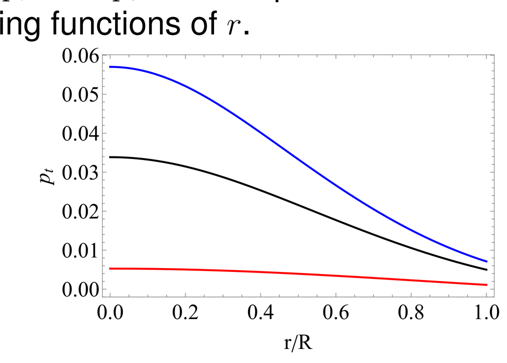

Figure 6: Tangential pressure for different values of K

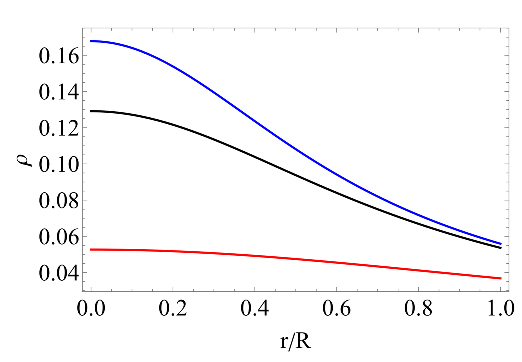

Figure 8: Density for different values of K

5. **Energy Conditions:**

Dominant Energy Condition (DEC): $\rho \ge p_r, \quad \rho \ge p_t,$ (25)
Strong Energy Condition (SEC): $\rho \ge p_r + 2p_t.$ (26)

Satisfaction of the SEC is desirable for ensuring equilibrium stability.

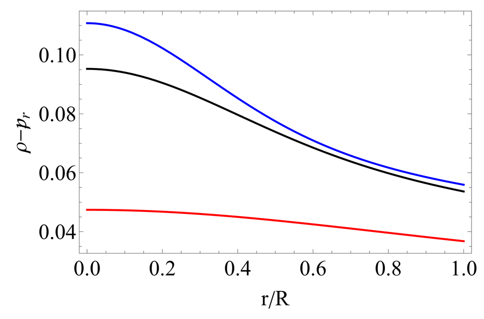

Figure 9: Dominant Energy Condition for different values of K

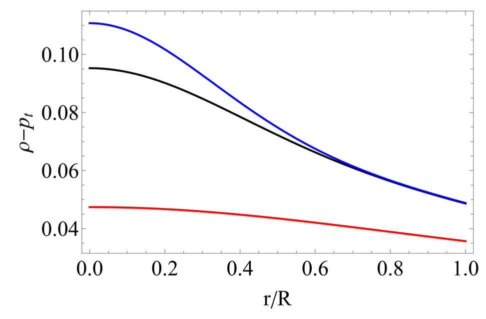

Figure 10: Dominant Energy Condition for different values of K

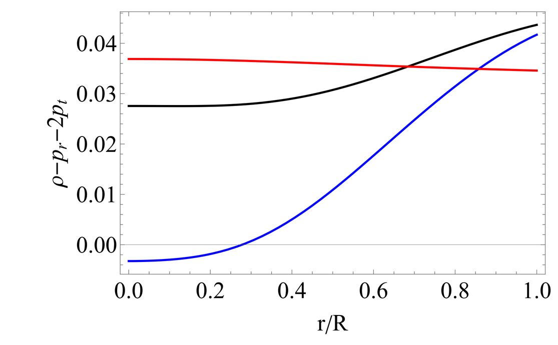

Figure 12: Strong Energy Condition for different values of K

6. **Gravitational Redshift:**

$$Z(r) = \frac{1}{\sqrt{e^{\nu(r)}}} - 1, \tag{27}$$

must be continuous, positive, and decrease with radius. Its surface value must obey the upper bound $Z(R) < 5.211$, and if the model satisfies the SEC, the limit becomes more restrictive $Z(R) < 3.842$.

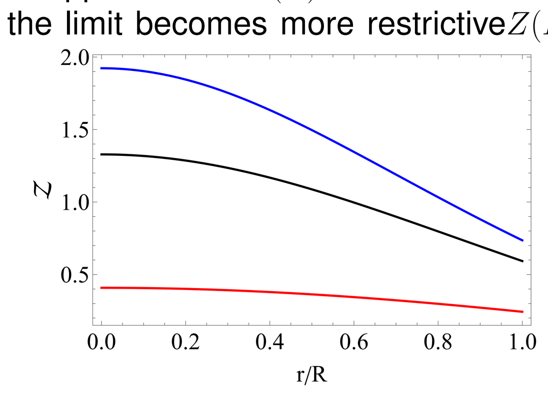

Figure 14: Gravitational Redshift for different values of K

5. Energy Exchange Between Relativistic Fluids

The results indicate that $\Delta E > 0$, with the energy exchange increasing in the outer regions of the compact star. This suggests that the polytrope transfers energy to the environment to coexist with the perfect fluid.

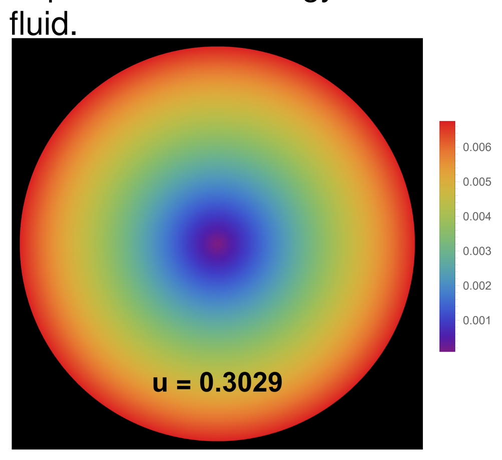

Figure 15: ΔE density plot for K = 0.43

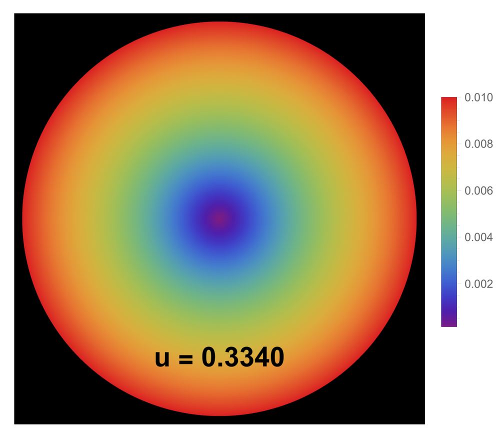

Figure 16: ΔE density plot for K = 0.44

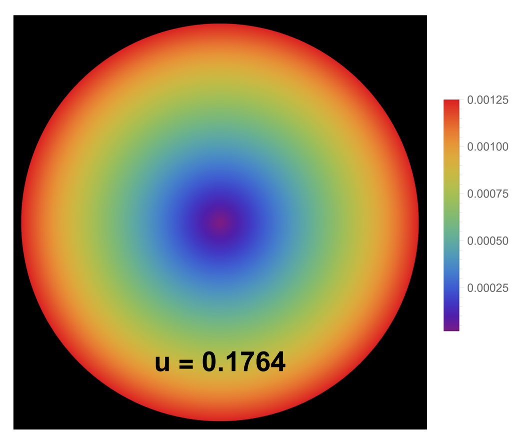

Figure 18: ΔE density plot for K = 0.47

References

[1] Albert Einstein. Die feldgleichungen der gravitation. *Annalen Der Physik*, 1915.

[2] Albert Einstein. Die grundlage der allgemeinen relativitätstheorie. *Annalen Der Physik*, 1916.

[3] G. P. Horedt. *Polytropes: Applications in Astrophysics and Related Fields*. Astrophysics and Space Science Library. Springer Dordrecht, 1 edition, 2004.

[4] B. V. Ivanov. Analytical study of anisotropic compact star models. *The European Physical Journal C*, 77(11):738, 09 2017.

[5] J. Ovalle. Searching exact solutions for compact stars in braneworld: A conjecture. *Modern Physics Letters A*, 23(38):3247–3263, 2008.

[6] J. Ovalle. Decoupling gravitational sources in general relativity: The extended case. *Physics Letters B*, 788:213–218, 2019.

[7] J. Ovalle, E. Contreras, and Z Stuchlik. Energy exchange between relativistic fluids: the polytropic case. *Eur. Phys. J. C*, 2022.

[8] Jorge Ovalle. Decoupling gravitational sources in general relativity: from perfect to anisotropic fluids. *Physical Review D*, 95(10):104019, 2017.

[9] Max Wyman. Schwarzschild interior solution in an isotropic coordinate system. *Physical Review*, 70(1-2):74 – 76, 1946. Cited by: 12.